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LETTER TO THE EDITOR

The origin of the ultrametric topology of spin glasses

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Abstract. We discuss the $N\to 0$ limit of a variety of field theories, in particular those describing spin glasses. We show how the convergence is naturally described using a p-adic norm. Replica symmetry breaking and combinatorial calculations in these theories make sense mathematically in the p-adic regime. The p-adic order parameter is viewed naturally as a topological central extension related to holonomy (like Berry's phase) from the quenched frustration in spin glasses.

The replica trick of field theories [1] and the consequent $N\to 0$ limit has been very useful in a variety of systems, in particular, spin glasses (for excellent reviews see [2, 3]). The main idea is to obtain a functional representation for $\ln Z$, Z being the partition function, by using the identity $\ln Z = \lim_{N\to 0}(Z^N - 1)/N$. One defines Z^N by introducing N fields (or spins) in the standard functional integral sums. One can average over coupling constants and then set $N=0$ at the end of the calculation. All physical observables of the system can be calculated this way.

The picture that emerges in the case of a spin glass is characterised by the existence of a large number (infinite in the infinite-volume limit) of equilibrium states that are almost degenerate, i.e. the free-energy valleys are separated by free-energy barriers that become infinitely high in the thermodynamic limit. Essential to the description of the thermodynamics is Parisi's proposal [4] of an order parameter that breaks the permutation symmetry of the replicas.

There is, however, a puzzling feature of Parisi's proposal. What is the meaning of breaking the permutation symmetry, P_N, in the $N\to 0$ limit?

In this letter we shall clarify this puzzle by using a formulation of the spin-glass problem in terms of p-adic norms and p-adic numbers. We claim that the mystery generated by the $N\to 0$ limit is a result of our intuition being accustomed to considering limits with respect to the absolute value norm, the norm that is considered when completing the rational numbers to construct the real numbers. By Ostrowski's theorem, there exist an infinite family of p-adic norms, one for each prime p, and only these alternative norms, with respect to which one can complete the rational numbers to form the p-adic numbers (see [5] for two excellent books on p-adic numbers and analysis). Given a prime p and a rational number x/y, with x and y relatively prime, then x/y can be expressed as $x/y = p^L a/b$ with L an integer equal to the largest power of p dividing x minus the largest power of p dividing y (either or both will be zero). Then the p-adic norm of x/y, $|x/y|_p = p^{-L}$. The p-adic numbers Q_p are obtained by completing the rationals in the p-adic norm, while the p-adic integers Z_p are the set of elements in Q_p with norm ≤ 1. The p-adic $|N|_P\to 0$ limit is obtained by considering

$N = p^L$ (L positive and integral) replicas with $L \to \infty$. The notion of a large permutation symmetry broken by an order parameter described by a huge matrix now makes sense, both intuitively and logically. For p^L states, we will see that the order parameter can take only L different values just like the p-adic norm. Just as the absolute value of ϕ equals the absolute value of $-\phi$, the p-adic norms of $0, 1, \ldots, p-1$ are all the same. Spontaneous symmetry breaking in the usual sense corresponds to choosing ϕ or $-\phi$ as a vacuum expectation value of the field if $\phi = 0$ and perturbing about this choice of vacuum. Replica symmetry breaking involves a choice of $0, 1, \ldots, p-1$ and perturbing by looking at values of the order parameter to order p^2 with norm p^{-2}, i.e. the system has a natural small parameter, p. Moreover, the ultrametric description of states follows immediately from the ultrametric topology of the p-adics $|x+y|_p \leqslant \max(|x|_p, |y|_p)$. Equivalently, every p-adic triangle is isosceles since if $|x|_p < |y|_p$, the highest power of p that divides $x-y$ must be the same as that that divides y since the highest power of p that divides x is even greater. Even though the limit is taken p-adically, physical observables can be related to rational numbers. In particular, the free energy is written as a p-adic integral. Finally, one might ask what picks a particular prime. One could consider each prime in turn by localising with respect to each prime, i.e. only considering factors of a particular prime, and allow for ultrametric topologies with varying branching numbers. One returns to the reals by means of an adelic relation which generalises the fact that $\Pi_p |x|_p = 1$ (the product over all the primes, including the prime at infinity corresponding to the absolute value) for x rational. For example, the self-avoiding random walk [6] can be considered over all places or norms including the ∞ real and complex ones, with $|\,|_\infty$ the absolute value. An unanswered question is why some theories realise replica symmetry breaking and others do not.

The mean-field approximation to the partition function has been written in terms of a sum of N Ising spins S_i interacting in pairs i, j through couplings J_{ij} that are random, obeying a symmetric probability distribution with variance $1/\sqrt{N}$:

$$Z(\beta, J, h) = \sum_S \exp(-\beta H(J, h, S)) \tag{1}$$

$$H(J, h, S) = -\sum_i J_{ij} S_i S_j - h \sum_i S_i. \tag{2}$$

Parisi's order parameter is given in terms of the local overlap of states. A pure equilibrium state α has local magnetisation $m_i^\alpha = \langle S_i^\alpha \rangle$, and the order parameter $q^{\alpha\beta} = (1/N) \sum_{i=1}^N m_i^\alpha m_i^\beta$ describes the overlap between the magnetisation of states α and β.

We claim in this letter that the convergence of the $N \to 0$ limit can be discussed consistently in the p-adic sense, that the replica symmetry breaking order parameter is naturally described as a function on the p-adics, and that the distribution function for clusters of states can be computed in terms of p-adically interpolated functions [5]. The notion of p-adic interpolation and measure theory follows from the fact that rational integers are dense in the p-adic integers over which one can define a measure and integrate functions.

A simple example of convergence in the p-adic domain is the fact that for k a positive integer

$$\lim_{N \to 0} \frac{1}{N} S_k(N) = B_k \tag{3}$$

$$S_k(N) = \sum_{i=1}^{N-1} i^k \tag{4}$$

where B_k is the kth Bernoulli number. This follows from the fact that

$$\sum_{i=0}^{N-1} e^{ix} = \frac{e^{Nx}-1}{e^x-1}$$

$$= \frac{e^{Nx}-1}{x} \frac{x}{e^x-1} \tag{5}$$

$$= \sum \frac{1}{k!} x^k S_k(N) = \frac{e^{Nx}-1}{x} \sum \frac{B_k x^k}{k!}. \tag{6}$$

However, the convergence makes sense p-adically because we are taking $N \to 0$ in a sum up to $N-1$ in (4). In fact, simple estimates show this convergence [5].

Similarly, we want to consider the p-adic convergence of $(Z^N-1)/N$. We first assume that we are in a finite system with integer-valued coupling constants. We define the convergence of Z^N to a p-adic number, and take the p-adic norm of $(Z^N-1)/N$ to obtain a real number, i.e. a measurable quantity. In principal, this quantity should be independent of the prime as a result of some adelic relation similar to the one stating that $\Sigma_p \log|x|_p = 0$ for x rational. (Note that the adelic relation is different from the one stated because we need here to take the norm of the logarithm, not the logarithm of the norm.) The partition function can now be defined p-adically for $\beta = \ln t$ and t taking appropriate values modulo problems of the infinite-volume limit to be addressed later. This is based on the theorem that the power function n^S can be defined p-adically if $n = 1 \bmod p$ so that $n^s - n^{s'}$ is small p-adically when $s-s'$ is small p-adically, i.e. divisible by a large in the absolute sense $s-s' = ap^N$, $a \in Z/pZ$

$$|n^s - n^{s'}|_p = |n^s||1-n^{s-s'}|_p = |n^s||1-n^{p^N}|_p \leqslant |n^s|p^{-L}. \tag{7}$$

We can schematically show convergence to a p-adic number for identical replicas so that Z^N is proportional to Z^{p^L}. Therefore for $t = 1 \bmod p$, $t = 1+pa$,

$$\left|\frac{t^{p^L}-1}{p^L} - \frac{t^{p^M}-1}{p^M}\right|_p \leqslant \left|\left(\frac{p^L-p^M}{2}\right)a^2p^2\right|_p = p^{-L-2}\left|\frac{a^2}{2}\right|p \qquad p \neq 2. \tag{8}$$

We can ignore the factors of volume because one can always choose the volume equal to l^{2n} where l is prime to p so that $|l|_p = 1$.

The hierarchical organisation of the spin-glass states is naturally described in a p-adic framework for a fixed prime-p branching of states. In particular, for a system where the Parisi replica symmetry breaking heirarchy arises, the hierarchy can be organised along a p-adic tree. At each level, the permutation symmetry, P_{p^L} breaks further, first to $P_{p^{L-1}} \times P_p$, then to $P_{p^{L-2}} \times P_p \times P_p$ and continues L times until the symmetry is broken as far as the ultrametric topology will allow. For the p^Lth replica, there are $2p^L-1$ possible value of the magnetisation. For a finite number of states α, the order parameter $q^{\alpha\beta}$ is labelled by a pair of integers α, β which designate points on a tree. The magnetisation can be described in terms of a locally constant function $m(x)$ of the p-adic coordinate, x. For example, it may be x itself since $|x|_p \leqslant 1$ for x in Z_p. Finally, the overlap is determined by $[m(x)-m(y)]^2$. Once more we note that there is a p-adic interpolation, this time of the measure on the replica space. For any L, the p^Lth stage of replica symmetry defines a measure which can be interpolated to $L \to \infty$, i.e. on the p-adic integers. This can be used to define the characteristic function

$$g(y) = \frac{1}{p^L(p^L-1)} \sum_{\alpha=0}^{p^L-1} \sum_{\beta=0}^{p^L-1} \exp(yq^{\alpha\beta}) \xrightarrow[L\to\infty]{} \int_{Z_p} d\mu_Z(x) \exp(yq(x)) \tag{9}$$

(note that there is an unexplained sign difference with (7) because $\lim_{n\to 0}(1/n-1)=-1$) as well as generalised moments with

$$\begin{aligned}\mu_Z(U(a,N)) &= Z^a/1-Z^{p^N}\\ U(a,N) &= \{b|b=a \bmod p^N\}=\{b||b-a|_p\leqslant p^{-N}\}.\end{aligned}\tag{10}$$

An important question to which we shall return remains. Namely, why does the order parameter take only L values?

One can use p-adic interpolation to compute distribution properties of clusters of spin-glass states [7]. For example, consider the probability M_k of choosing k different replicas in clusters of size m:

$$M_k=\frac{n(m-1)\ldots(m-k+1)}{n(n-1)\ldots(n-k+1)}\tag{11}$$

$$\xrightarrow[\substack{n\to 0\\ m\to 1-y}]{}\frac{\Gamma(-1+k+y)}{\Gamma(k)\Gamma(y)}\tag{12}$$

which is like the inverse Veneziano amplitude in open string theory. This formula can be p-adically interpolated in terms of the p-adic Morita gamma function [8]

$$\Gamma_p(x)=\lim_{k\to x}(-1)\prod_{p=i}^{k-1} i\tag{13}$$

or in terms of the real-valued Euler factor

$$\Gamma(x)=\int_{Z_p}\mathrm{d}\lambda_p|\lambda_p|_p^x\tag{14}$$

$$=1/(1-p^{-x})\tag{15}$$

where λp is the Haar measure on the units in Z_p.

To construct a model, we consider a curve or projective line over a finite field like Z/pZ. For example, a projective line over Z/pZ has $p+1$ points: $(x,y)\sim\lambda(x,y)$ such that $x, y, \lambda\in Z/pZ^*$, the non-zero integers mod p. One can think of each point as the start of a p-adic expansion, so that mod p^2 each point has p elements sitting over it, mod p^3 each has p^2 elements, etc. This naturally gives rise to the type of replicas one might consider in the $N\to 0$ limit. Moreover, on the projective line, each point is equivalent so that long-range interactions are expected. These interactions become infinitely long as $p\to\infty$.

Having understood logically the origin of the $N\to 0$ limit, one might then ask why the order parameter takes only L values when the spins take $2p^L-1$ values. In answer to this, we suggest that the entire phenomenon of spin glasses resulting from non-ergodicity of the $N\to 0$ limit can be perceived as resulting from p-adic holonomy or Berry's phase [9] where the order parameter classifies the type of holonomy. Recall that Berry's phase results from adiabatic transport in a theory described as a non-trivial line bundle [10]. The theory behaves as if it had a magnetic monopole in parameter space. In a spin glass, the coupling J_{ij} can be frustrated and therefore, gives rise to a non-trivial line bundle. The order parameter acts like the monopole or curvature in a parameter space which classifies the line bundles. Transport of a spin around a loop or along a line in physical space can result in a rotation in p-adic parameter space. This destroys ergodicity. Order parameters with arbitrary numbers of indices describe additional topological invariants of the bundle.

To be more precise, the infinite-volume limit of the replicated theory results in a Morse theory, i.e. the free energy is determined by a stationary phase with critical points given by the solutions [1, 3, pp 866, 867] of

$$q^{\alpha\beta} = \langle S^\alpha S^\beta \rangle \qquad m^\alpha = \langle S^\alpha \rangle \qquad S^\alpha - \frac{1}{N}\sum_i S_i^\alpha. \tag{16}$$

If we introduce a fictitious time on which quantum spins depend [11], $q^{\alpha\beta}$ can be understood in terms of a p-adic height or intersection of the S^α. (After all, overlaps are really intersections.) If the fictitious time describes evolution in a stochastic quantisation scheme the p-adic limiting distribution of states is not ergodic and does not approach the Gibbs distribution ($\exp Ep^N \to 1$ for all E). This gives a central extension to the classically commuting algebra of spins. The central extension is equivalent to the way a Green function on a Riemann surface gives an intersection or height function in terms of which one can compute a central extension of the algebra of meromorphic differentials on the Riemann surface [12]. Therefore quantum mechanically $q^{\alpha\beta}$ is a quadratic form which gives a central extension which descends from a topological invariant on a projective space of p-adic spins. Just as one can define CP^1 as the set of two complex numbers (Z_1, Z_2) modulo a non-zero complex number λ, one can define a p-adic projective space for spins S^α. Moreover, there is a natural line bundle over CP^1 determined by the winding of the phase of λ, which has a p-adic analogue. $q^{\alpha\beta}$ describes the possible line bundles. This answers the question of why the order parameter $q^{\alpha\beta}$ takes only L possible values in a space of p^L spins. The infinite-volume limit is Morse theory with a p-adic height as a Morse function just as the absolute value height on CP^N classifies line bundles. Moreover, the instability of the ansatz that $q^{\alpha\beta}$ is constant appears as a consequence of the existence of negative eigenvalues in the Morse theory.

Finally, we would like to mention that there is a Hecke algebra of spin correspondences [13] if and only if the order parameter is given in terms of the p-adic height function. Assume a general order parameter $q^{\alpha\beta}$. Let Θ_n denote the correspondence which associate with each spin S_α the formal sum of spin S_β such that $q^{\alpha\beta} = n$. Put $T_0 = \theta_0 =$ indentity. $T_1 = \theta_1$, $T_n = \theta_n + T_{n-2}$, then $T_1 T_n = T_{n+1} + pT_{n-1}$ if and only if $q^{\alpha\beta}$ is the p-adic height. Moreover, $\Sigma T_n x^n = (1 - T_1 x + px^2)^{-1}$. This suggests the possibility of using additional methods of string theory and automorphic forms. In particular, the existence of a Hecke algebra can lead to irreducible group representations of a p-adic Lie group like $\mathrm{SL}(2, Z_p)$ [13]. The eigenfunctions of the Hecke algebra then form a basis of states whose Mellin transform gives an L function with a Euler product expansion analogous to the Euler product for a partition function in string theory. In the mean-field approximation for spin glasses, the partition function should have an analogous Euler function expansion. The rigid structure of the vacua labelled by the order parameters is reflected in the rich structure of the representation theory analogous to the way instantons give θ vacua and translation symmetry in crystals gives rise to Bloch states [14].

We do not want to leave the reader with the impression that the above discussion is mere philosophy. As an example, we consider a theory whose moduli space is determined by the Fermat curve (for an excellent discussion of Fermat curves, see [15]):

$$V = \{(x, y) | x^N + y^N = 1, xy \neq 0\}. \tag{17}$$

This will lead, through the Weil zeta function, to an exactly solvable model in the $N = p^L \to \infty$ limit, i.e. the $|N|_p \to 0$ limit, that is presented on a tree. (Incidentally, the higher-dimensional Fermat curves depending on more variables are very useful for higher-dimensional moduli spaces.) The Fermat curve has a $G=\mu^N \times \mu^N$ symmetry, where μ^N is the mutiplicative group of the Nth roots of unity. The possible diffusion constants on the tree represent the possible values of an order parameter that can break this permutation symmetry.

The Weil zeta function is the partition function and can be evaluated for this curve over the finite field F_q with $q=1 \pmod N$ so that μ^N lies in F_q. If F is the Frobenius map $x \to x^q$, we have that

$$\exp\left(\sum_{n=1}^{\infty} \frac{T^n}{n} \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})|\mathrm{fix}(F^{n}\circ g)|\right) = \frac{\det(1-TF^*|H^1(V))}{(1-T)(1-qT)}. \tag{18}$$

The characters χ determine the energy. We can choose

$$\chi = \chi_1\chi_2 \qquad \chi_i : x \to \omega(x)^{a_i} \qquad i=1,2 \tag{19}$$

with ω a Teichmuller character that imbeds μ^N in K, an extension of Q_p by $Q_p(1^{1/N})$. Finally, if

$$\chi_i : x \to x_i(x^{(q-1)/d}) \tag{20}$$

then the eigenvalues of Frobenius are

$$-J(\chi_1, \chi_2) = \sum_{\substack{u \in F_q^* \\ u \neq 0,1}} \tilde{\chi}_1(u)\tilde{\chi}_2(1-u) \tag{21}$$

which can be explicitly evaluated in terms of a p-adic beta function. We can now put a tree at each spacetime point and average over a distribution for T, treated as a spacetime coupling. In the infinite-volume limit this can break the replica symmetry G that permutes the characters $\chi_r\chi_s$ with $0<r$, $s<N$, $r+s \neq N$. We must evaluate

$$\prod_{ij} \int \mathrm{d}T_{ij} P(T_{ij}) \det(1-T_{ij}F|H^1) \tag{22}$$

in the standard way [2, 3] to obtain the breaking of replica symmetry with p-adic order parameters in the $|N|_p \to 0$ limit. The order parameters are like Berry phases insofar as they are related to the periods of the Fermat curves. This theory is independent of p, up to the consideration $q=1 \pmod N$.

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